Painlevé VI, hypergeometric hierarchies and Ward ansätze

2006 J. Phys. A: Math. Gen. 39 12265

(http://iopscience.iop.org/0305-4470/39/39/S17)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 12265–12269 doi:10.1088/0305-4470/39/39/S17

Painlevé VI, hypergeometric hierarchies and Ward ansätze

M R Shah and N M J Woodhouse

Mathematical Institute, 24-29 St Giles', Oxford, OX1 3LB, UK

E-mail: shahm@maths.ox.ac.uk and nwoodh@maths.ox.ac.uk

Received 30 May 2006
Published 13 September 2006
Online at stacks.iop.org/JPhysA/39/12265

Abstract
We show that Okamoto's classical solutions to P_{VI} constructed from a seed solution of Gauss' hypergeometric equation can be derived very simply from the Ward ansätze for ASDYM connections.

PACS numbers: 02.30.Gp, 02.30.Ik, 02.40.Tt
Mathematics Subject Classification: 34M55, 32L25

1. The anti-self-dual Yang–Mills equation and P_{VI}

Consider complexified Minkowski space $\mathbb{CM}$, the four complex dimensional space with complex local coordinates $(w, z, \tilde{w}, \tilde{z})$ and metric

$$ds^2 = 2(dz \otimes d\tilde{z} - dw \otimes d\tilde{w}).$$

The *anti-self-dual Yang–Mills* (ASDYM) equation is the condition on an sl(2, $\mathbb{C}$) connection $d + \Phi$ that its curvature 2-form should satisfy $F = -\star F$; alternatively, that the two operators

$$L = \zeta(\partial_w + \Phi_w) - \partial_{\tilde{z}} - \Phi_{\tilde{z}} \qquad M = \zeta(\partial_z + \Phi_z) - \partial_{\tilde{w}} - \Phi_{\tilde{w}}$$

should commute for any value of the complex parameter ζ. The sixth Painlevé equation (P_{VI}) is equivalent to the reduction of this system under the conformal symmetries generated by the three conformal Killing vectors

$$X_1 = -z\partial_z - w\partial_w, \qquad X_2 = -\tilde{w}\partial_{\tilde{w}} - \tilde{z}\partial_{\tilde{z}}, \qquad X_3 = z\partial_z + \tilde{w}\partial_{\tilde{w}}$$

(Mason and Woodhouse 1996, section 7.4). We can make explicit the connection between the ASDYM equation and P_{VI} through the corresponding isomonodromy problem. If the connection is invariant, then, after a gauge transformation, Φ can be written in the form

$$\Phi = A\frac{dw}{w} + B\frac{dz}{z} + \tilde{A}\frac{d\tilde{w}}{\tilde{w}} + \tilde{B}\frac{d\tilde{z}}{\tilde{z}}, \qquad (1)$$

where the matrices $A, B, \tilde{A}, \tilde{B}$ are functions of $t = z\tilde{z}/w\tilde{w}$ alone. In this case, the ASDYM equation is equivalent to the compatibility of the five linear equations

$$Lg = 0, \qquad Mg = 0, \qquad X_1 g - \zeta\partial_\zeta g = 0, \qquad X_2 g + \zeta\partial_\zeta g = 0, \qquad X_3 g = 0$$

in the unknown $g(w, z, \tilde{w}, \tilde{z})$ with values in $\mathrm{SL}(2, \mathbb{C})$. A solution then determines a flat meromorphic connection $\mathrm{d} - (\mathrm{d}g)g^{-1}$ on a bundle over $\mathbb{CM} \times \mathbb{CP}_1$. By picking out the $\mathrm{d}\zeta$-component of $\mathrm{d}g$, we see that, at a fixed point of Minkowksi space, g is the fundamental solution of the Fuchsian system

$$\frac{\mathrm{d}g}{\mathrm{d}\zeta} = \left(\frac{A + \tilde{B}}{\zeta + r} + \frac{B + \tilde{A}}{\zeta + s} - \frac{\tilde{A} + \tilde{B}}{\zeta} \right) g$$

where $r = w/\tilde{z}$ and $s = z/\tilde{w}$. This has four regular singularities, at $-r, -s$, 0 and ∞, with cross-ratio $t = s/r$; as t varies the system is deformed isomonodromically, and so we obtain a solution of the Schlesinger equations, and hence of $\mathrm{P_{VI}}$. The Painlevé parameters are expressed in terms of four invariants of the deformations, given by the traces of

$$(A + \tilde{B})^2, \qquad (B + \tilde{A})^2, \qquad (A + B)^2, \qquad (\tilde{A} + \tilde{B})^2.$$

2. Ward ansätze and wave equations

The identification of $\mathrm{P_{VI}}$ with a reduction of the ASDYM equation opens the door to the construction of solutions by twistor methods. Elsewhere, we shall show how this method generalizes in a very straightforward way to the general isomonodromy problem.

In the twistor construction, solutions of the ASDYM equation correspond to holomorphic bundles over a neighbourhood of a line in $\mathbb{CP}_3$. The bundle is required to have a trivial restriction to the line. It can be characterized by its $\mathrm{SL}(2, \mathbb{C})$-valued patching matrix T between trivializations over two open sets, V_0, V_∞, whose intersection meets the line in an annulus. In operational terms, the solution is recovered by expressing $T = T(\lambda, \mu, \zeta)$ in terms of inhomogeneous coordinates λ, μ, ζ on $\mathbb{CP}_3$ and by finding a Birkhoff factorization

$$T(w + \zeta\tilde{z}, z + \zeta\tilde{w}, \zeta) = H_\infty H_0^{-1}$$

for each fixed spacetime point labelled by $w, z, \tilde{w}, \tilde{z}$. With $h_0 = H_0|_{\zeta=0}$ and $h_\infty = H_\infty|_{\zeta=\infty}$, the gauge potential is given by

$$\Phi_{\tilde{z}} = h_0^{-1}\partial_{\tilde{z}} h_0, \qquad \Phi_{\tilde{w}} = h_0^{-1}\partial_{\tilde{w}} h_0, \qquad \Phi_w = h_\infty^{-1}\partial_w h_\infty, \qquad \Phi_z = h_\infty^{-1}\partial_z h_\infty.$$

Thus the twistor method reduces the solution of the ASDYM equation to a Riemann–Hilbert problem (Ward and Wells 1990, section 8). In general, this is intractable, but the splitting is known if the twistor bundle is the extension of a line bundle $\mathcal{O}(k)$ by another, $\mathcal{O}(-k)$, the Ward ansatz (Ward 1981). In this case, the transition matrix is

$$T_k = \begin{pmatrix} \zeta^k & \phi \\ 0 & \zeta^{-k} \end{pmatrix},$$

where k is a non-negative integer. The entries are functions of λ, μ, ζ; when they are expressed in terms of $w, z, \tilde{w}, \tilde{z}, \zeta$, they are therefore constant along the vector fields

$$l = \zeta\partial_w - \partial_{\tilde{z}}, \qquad m = \zeta\partial_z - \partial_{\tilde{w}}.$$

Hence, if ϕ has Laurent expansion

$$\phi = \sum_{i=-\infty}^{\infty} \phi_i \zeta^i,$$

then the coefficients ϕ_i must satisfy

$$\frac{\partial\phi_i}{\partial w} = \frac{\partial\phi_{i+1}}{\partial\tilde{z}}, \qquad \frac{\partial\phi_i}{\partial z} = \frac{\partial\phi_{i+1}}{\partial\tilde{w}}.$$

Therefore, each ϕ_i satisfies the complex wave equation

$$\frac{\partial^2\phi_i}{\partial w\partial\tilde{w}} = \frac{\partial^2\phi_i}{\partial z\partial\tilde{z}}.$$

Our aim is to find suitable ϕ_i starting from a single seed solution to Gauss' hypergeometric equation, and then let k label members of a hierarchy of solutions to P_{VI} derived from it together with solutions of contiguous equations. By direct calculation, we have the following lemma.

Lemma 1. *Let $t = z\tilde{z}/w\tilde{w}$, and let $'$ denote differentiation with respect to t. Then*

$$\psi = \frac{\tilde{z}^{c-1}y}{w^a\tilde{w}^b}$$

satisfies the wave equation if and only if y satisfies the hypergeometric equation with parameters (a, b, c); that is

$$t(1-t)y'' + (c - (a+b+1)t)y' - aby = 0.$$

If $\hat{\psi} = \tilde{z}^{\hat{c}-1}\hat{y}/w^{\hat{a}}\tilde{w}^{\hat{b}}$, with $\hat{a} = a+1$, $\hat{b} = b$, $\hat{c} = c+1$, then the equations

$$\frac{\partial\psi}{\partial w} = \frac{\partial\hat{\psi}}{\partial\tilde{z}}, \qquad \frac{\partial\psi}{\partial z} = \frac{\partial\hat{\psi}}{\partial\tilde{w}}$$

are compatible if and only if y satisfies the hypergeometric equation with parameters (a, b, c). In this case, $\hat{y}$ satisfies the hypergeometric equation with parameters $(\hat{a}, \hat{b}, \hat{c})$ and

$$-b\hat{y} - t\hat{y}' = y', \qquad c\hat{y} + t\hat{y}' = -ay - ty'.$$

By starting with a given a solution y to the hypergeometric equation with parameters (a, b, c), we can construct a Laurent series for ϕ with the required properties, with the coefficients

$$\phi_0 = \psi, \qquad \phi_{i+1} = \hat{\phi}_i.$$

Each stage involves integration, and so ϕ and the transition matrix are not uniquely determined by the seed. However, the holomorphic bundle and the gauge-class of Φ are independent of the choices made. The ϕ_i $(-\infty < i < \infty)$ are expressed in terms of a sequence of hypergeometric functions y_i with parameters (a_i, b_i, c_i), which are related by

$$-b_i y_{i+1} - ty'_{i+1} = y'_i, \qquad c_i y_{i+1} + ty'_{i+1} = -a_i y - ty'_i,$$

where $a_i = a + i$, $b_i = b$ and $c_i = c + i$.

We reconstruct the Yang–Mills field Φ by following the steps in Ward and Wells (1990, section 8.2). Put

$$Y = \begin{pmatrix} y_{k-1} & y_{k-2} & \cdots & y_0 \\ y_{k-2} & \ddots & & y_{-1} \\ \vdots & & \ddots & \vdots \\ y_0 & y_{-1} & \cdots & y_{1-k} \end{pmatrix}.$$

Then the matrix M in theorem 8.2.2 of Ward and Wells (1990) is

$$M = \frac{z^{c+k-2}}{w^{a+k-1}\tilde{w}^b} DYD,$$

where D is the diagonal matrix with diagonal entries $1, r, r^2, \ldots, r^{k-1}$. So we can read off the corresponding Yang–Mills field from the theorem. After a gauge transformation by

$$\begin{pmatrix} r^{k/2} & 0 \\ 0 & r^{-k/2} \end{pmatrix},$$

it is in the form (1). We put

$$e = (Y^{-1})_{11}, \qquad f = (Y^{-1})_{1k}, \qquad g = (Y^{-1})_{kk}.$$

Then

$$A = \frac{1}{2f}\begin{pmatrix} tf' + (k-a)f & -2te' + 2(k+c-2)e \\ 0 & -tf' + (a-k)f \end{pmatrix}$$

$$B = \frac{1}{2f}\begin{pmatrix} -tf' & 2t^2e' - 2bte \\ 0 & tf' \end{pmatrix}$$

$$\tilde{A} = \frac{1}{2f}\begin{pmatrix} bf - tf' & 0 \\ -2g' & tf' - bf \end{pmatrix}$$

$$\tilde{B} = \frac{1}{2f}\begin{pmatrix} tf' + (1-c-k)f & 0 \\ 2tg' - 2(a-k+1)g & -tf' - (1-c-k)f \end{pmatrix}.$$

We then have a solution of Schlesinger's equations, and hence of P_{VI}. The corresponding parameters can be read off from the equivariance of the holomorphic bundle, and hence from the derivatives of T_k along $X_1 - \zeta\partial_\zeta$, $X_2 + \zeta\partial_\zeta$, X_3 and $X_4 = -X_1 - X_2 - X_3$. We have

$$X_1\phi - \zeta\partial_\zeta\phi = a\phi$$

and hence $X_1T_k - \zeta\partial_\zeta T_k = \theta_\infty T_k - T_k\theta_0$, where the θ are the constant diagonal matrices

$$\theta_\infty = \frac{1}{2}\begin{pmatrix} a-k & 0 \\ 0 & -a+k \end{pmatrix} \qquad \theta_0 = \frac{1}{2}\begin{pmatrix} a+k & 0 \\ 0 & -a-k \end{pmatrix}.$$

The twistor theory then implies that θ_∞ is conjugate to the Higgs field $A + B$, and hence that $\mathrm{tr}(A+B)^2 = \frac{1}{2}(a-k)^2$, as also follows directly from the expressions for A and B above. One similarly finds that

$$\mathrm{tr}(\tilde{A}+B)^2 = \tfrac{1}{2}b^2, \qquad \mathrm{tr}(A+\tilde{B})^2 = \tfrac{1}{2}(a-c+1)^2, \qquad \mathrm{tr}(\tilde{A}+\tilde{B})^2 = \tfrac{1}{2}(b-c+1-k)^2.$$

The solutions we have obtained here correspond to those in Okamoto (1987), expressed there in terms of determinants involving isomonodromic τ-functions. The general theory of hypergeometric solutions to P_{VI} is reviewed in Clarkson (2006).

3. Concluding remarks

The Ward ansätze are more general than those presented here. One can multiply the diagonal terms of the transition matrix by an arbitrarily chosen nonzero holomorphic function and its inverse, respectively. This has the effect of adding to the spacetime derivatives in the wave equation and the recursion relations the components of a 1-form

$$\alpha\frac{\mathrm{d}w}{w} + \beta\frac{\mathrm{d}z}{z} + \tilde{\alpha}\frac{\mathrm{d}\tilde{w}}{\tilde{w}} + \tilde{\beta}\frac{\mathrm{d}\tilde{z}}{\tilde{z}}$$

where $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ are constant. In this case the seed is a P-function (a solution of Riemann's differential equation). However, the P-functions that arise can be expressed in terms of hypergeometric functions, and the solutions of P_{VI} that one obtains are trivially related to the ones derived here.

One final remark is that the solutions to the ASDYM equation derived from the Ward ansätze are dense (in the Weierstrass sense) in the space of all solutions (Ivancovich *et al* 1990). However, the classical hypergeometric solutions are not dense in the solutions of P_{VI}. The reason is that although the ASDYM connection corresponding to a general solution of P_{VI} can be approximated to arbitrary accuracy by a connection derived from a Ward ansatz, this connection will not generally have exact invariance under the conformal transformations generated by X_1, X_2 and X_3. Nonetheless, the approximating connections are derived from the solution of linear equations (the wave equation and recursion relations), albeit ones in four independent variables, not one. This may provide a useful way to understand properties of general Painlevé transcendents.

References

Clarkson P A 2006 Painlevé equations—nonlinear special functions *Orthogonal Polynomials and Special Functions: Computation and Application (Lecture Notes in Mathematics* vol 1883) ed W van Assche and F Marcellan (Berlin: Springer) pp 331–411

Ivancovich J S, Mason L J and Newman E T 1990 On the density of the Ward ansätze in the space of anti-self-dual Yang–Mills solutions *Commun. Math. Phys.* **130** 139–55

Okamoto K 1987 Studies on the Painlevé equations: I. Sixth Painlevé equation P_{VI} *Ann. Mat. Pura Appl.* **146** 337–81

Mason L J and Woodhouse N M J 1996 *Integrability, Self-Duality and Twistor Theory (London Mathematical Society Monographs, New Series* vol 15) (Oxford: Oxford University Press)

Ward R S 1981 Ansätze for self-dual Yang–Mills fields *Commun. Math. Phys.* **80** 563–74

Ward R S and Wells R O 1990 *Twistor Geometry and Field Theory (Cambridge Monographs on Mathematical Physics)* (Cambridge: Cambridge University Press)